Problem 6

**Nice error bases**

| contact: | D. Schlingemann | solved by: | A. Klappenecker, M. Roetteler |
| date: | 08 Nov 2001 | last progress: | 28 May 2003 |

Version of 28 May 2003

For information about the QI open problems project at IMaPh refer to the web-pages [http://www.imaph.tu-bs.de/qi/problems/](http://www.imaph.tu-bs.de/qi/problems/). Please send questions, partial results or solutions concerning this problem to D. Schlingemann, email: d.schlingemann@tu-bs.de.

Please support us by suggesting further interesting problems!
Problem

There are two special constructions to obtain orthogonal bases of unitaries, i.e., collections of unitary operators \( U_i, \ i = 1, \ldots, d^2 \), on a \( d \)-dimensional Hilbert space, such that \( \text{tr}(U_i^* U_j) = d\delta_{ij} \).

On the one hand one can require in addition that the product of any two unitaries in the basis gives another one up to a phase, i.e., \( U_i U_j = \text{phase} \cdot U_k \). The composition of labels \((i, i) \mapsto k\) then defines a group, the “index group” of the basis. Bases of this kind have been called nice error bases.

On the other hand, one may require that, in a suitable basis of the Hilbert space, the unitaries are obtained as the products of a collection of \( d \) permutation operators and \( d \) multiplication operators. Bases constructed in this way are called of shift and multiply type.

The question that arises here is to decide whether every nice error basis is of shift and multiply type.

Background

Orthogonal bases are precisely what is needed to construct schemes for entanglement assisted teleportation or dense coding. For qubits (\( d = 2 \)) there is only one such basis up to left and right multiplication by fixed unitaries, namely the Pauli matrices together with the identity.

The shift and multiply constructions can be classified further: for the “shift” part one precisely needs a Latin square, whereas the multiplication part requires the construction of \( d \) complex Hadamard matrices.

A finite group \( H \) is called of central type if it possesses an irreducible representation in \( d = \sqrt{|H/Z(H)|} \) dimensions, where \( Z(H) \) is the center of \( H \).

Solution

An answer to the question, given above, has recently been found by Andreas Klappenecker and Martin Roetteler. They show in their article “On the monomiality of nice error basis” that there is in fact a nice error basis which is not of shift and multiplier type. Roughly their argumentation is based on the following: First one observes that every nice error basis which is of shift and multiplier type is monomial, i.e. each of its unitary matrices has in every row and column precisely one non-vanishing entry. An abstract error group is one which is generated by nice error bases (central extension of the index group). Such a group is of central type with cyclic center. Employing the theory of characters for these groups, which has been studied by P. Ferguson, I.M. Isaacs (see references given in [4]), an abstract error group can be constructed which has a non-monomial irreducible representation.
References


